Introduction to Mathematics and Modeling

lecture 1

Functions and Trigonometry

UNIVERSITY OF TWENTE.

academic year	:	18-19
lecture	:	1
build	:	November 13, 2018
slides	:	25

Part 2

GELGOOG Coffee Beans Husking Machine

1. Section 1.1: functions and their graphs
2. Section 1.2: combining and transforming functions
3. Section 1.3: trigonometry

- Monday : lecture LEC
  Tuesday/Thursday: assisted self-tuition AST
- Assisted self-tuition:

  Basic exercises : **B**
  Advanced exercises: **A**

- Three midterm tests and one resit. See MyTimeTable for date, time and location.

- Test 1: hand written test,
  Test 2: hybrid test (hand written test and MyLabsPlus test),
  Test 3: MyLabsPlus test.

- Examples and some exercises with *Mathematica*.

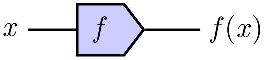
- Course schedule, slides and other materials can be found via the link on the Canvas page **Smart Environments (2018-1B)**, module **ItE: Introduction to Mathematics and Modeling I**, page **Lecture slides**.

Nr	Week	Topic
1	1	Basics: functions, graphs and trigonometry
2	2	Basics: the inverse; exponential functions and logarithms
		Midterm test 1
3	3	Differentiation: definition
4	4	Differentiation: rules and properties
5	5	Differentiation: applications
		Midterm test 2
6	6	Integration: definition and applications
7	7	Integration: the fundamental theorem; method of substitution
8	8	Integration: integration by parts
		Midterm test 3
		Resit

Definition

A **function** $f\colon D \to C$ *is a rule that assigns a unique element* $f(x)$ *in* C *to each element* x *in* D*.*

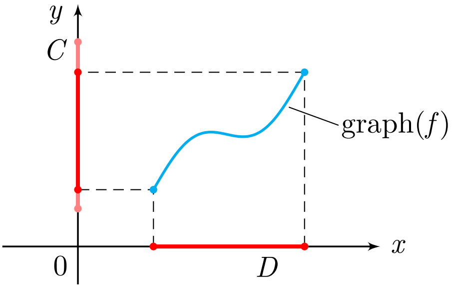
- The set D is the **domain** of f.
- The set C is the **codomain** of f.
- The **range** or **image** of f is the set of all function values $f(x)$.
- If f assigns y to x, then we denote this as $y = f(x)$ or $x \mapsto f(x)$.
- In a **diagram**:



Definition

Let D and C are subsets of $\mathbb{R}$. The **graph** *of a function $f\colon D \to C$ is defined as*

$$\operatorname{graph}(f) = \{(x, f(x)) \mid x \in D\}.$$



Vertical Line Test

A vertical line intersects the graph of a function in at most one point.

Mathematica

- *Defining a function:*

```
f[x_]:=1/Sqrt[x^2+1]
```

- *Plotting a function f with domain $[a, b]$:*

```
Plot[f[x],{x,a,b}]
```

Empty notebook (`Worksheet 1.nb`)

Definition

Let the function f be definied by a formula.

- *If the domain of f is not defined explicitly, then the domain consists of all numbers x for which $f(x)$ exists.*
- *If the codomain of f is not defined explicitly, then the codomain is chosen as large as possible.*

Example:

Let $f(x) = \sqrt{x-3}$.

- The expression $\sqrt{x-3}$ is defined for all x for which $x - 3 \geq 0$, hence

$$\mathrm{Dom}(f) = [3, \infty).$$

- The codomain is $\mathbb{R}$.
- The range is $[0, \infty)$.

Definition

Let $f\colon I \to \mathbb{R}$ be a function defined on an interval I.

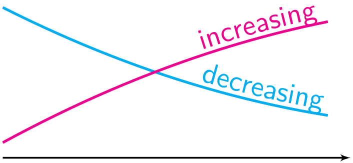
1. *The function f is* **increasing on** *I if for all $x_1, x_2 \in I$ with $x_1 < x_2$:*

$$f(x_1) < f(x_2).$$

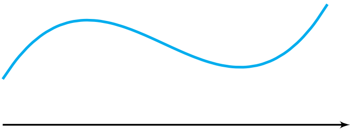
2. *The function f is* **decreasing on** *I if for all $x_1, x_2 \in I$ with $x_1 < x_2$:*

$$f(x_1) > f(x_2).$$

- A function that is decreasing or increasing is called **monotonous**.

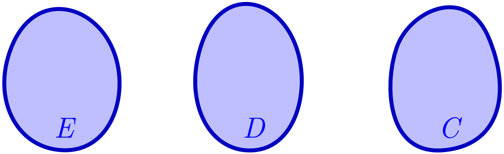


monotonous

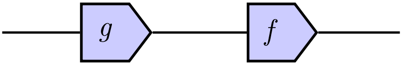


not monotonous

Addition:	$h(x) = f(x) + g(x)$	$f + g$
Subtraction:	$h(x) = f(x) - g(x)$	$f - g$
Multiplication:	$h(x) = f(x)g(x)$	$f\,g$
Division:	$h(x) = \dfrac{f(x)}{g(x)}$	$\dfrac{f}{g}$
Composition:	$h(x) = g(f(x))$	$g \circ f$

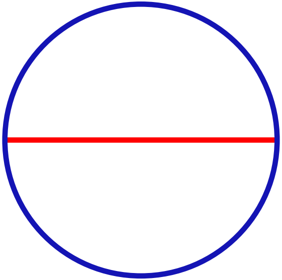


- Let $f\colon D \to C$ and $g\colon E \to D$ be two functions. The domain of f contains the range of g.
- The **composition of f and g** is defined as the function $f \circ g\colon E \to C$ that assigns the element $f(g(x))$ to every $x \in E$.
- Pronounce $f \circ g$ as "f after g".



(1) Define $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Find $f \circ g$,and $g \circ f$.

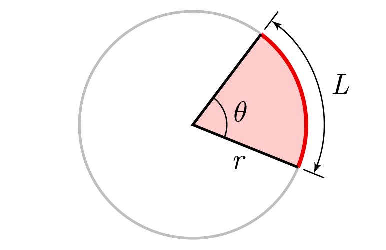
(2) Define $f(x) = \sqrt{x - 1}$ and $g(x) = x^2 + 2$. Find the domains and the ranges of f, g, $f \circ g$ and $g \circ f$.



$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

$$\pi \approx 3.141592653589793238462643383279502884197169399375105820\ldots$$

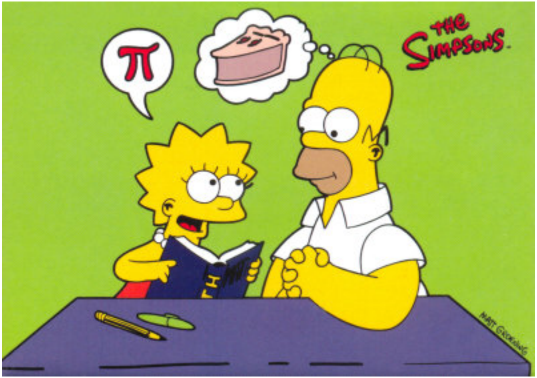
Theorem



In a sector, the length of the arc is equal to the product of the angle of the sector (measured in radians) and the radius of the circle.

$$L = r\theta$$

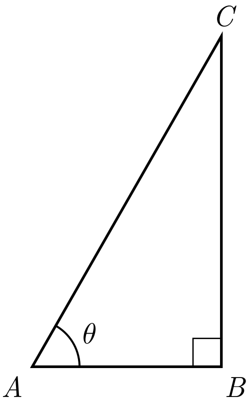
- The **radian** is a unit for measuring angles.
- One radian is approximately 57.3 degrees.
- A full circle is 2π radians.



The Simpsons, (TM) Twentieth Century Fox

$\pi =$ 

Triangle ABC is rectangular ($\angle ABC = \frac{\pi}{2}$), angle at A is acute ($0 < \theta < \frac{\pi}{2}$).

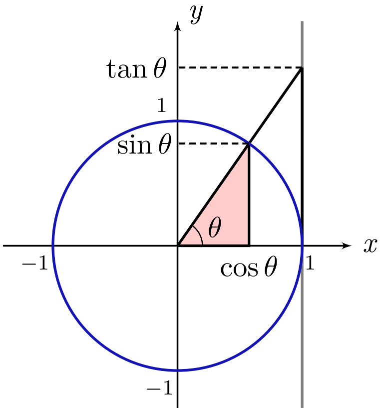


$$\cos\theta = \frac{AB}{AC},$$

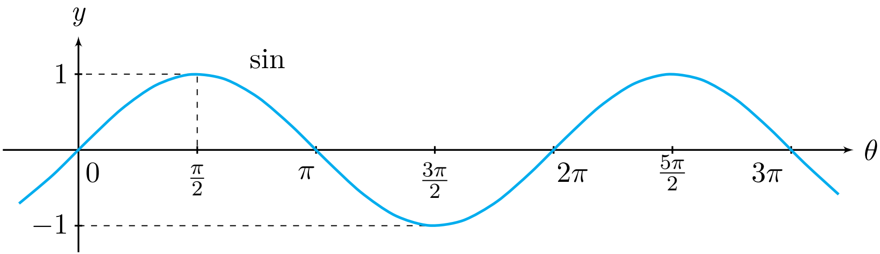
$$\sin\theta = \frac{BC}{AC},$$

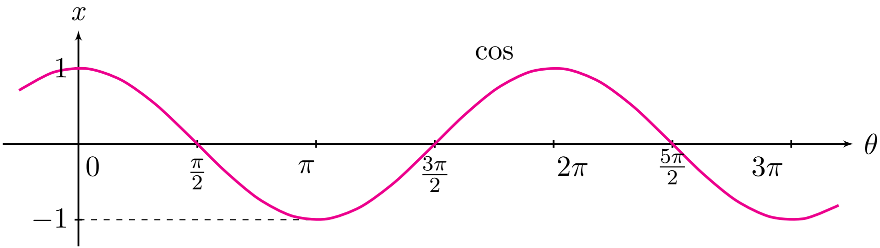
$$\tan\theta = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin\theta}{\cos\theta}.$$

For arbitrary angles, the sine, cosine are defined with the **unit circle**: the circle with center $(0, 0)$ and radius 1.

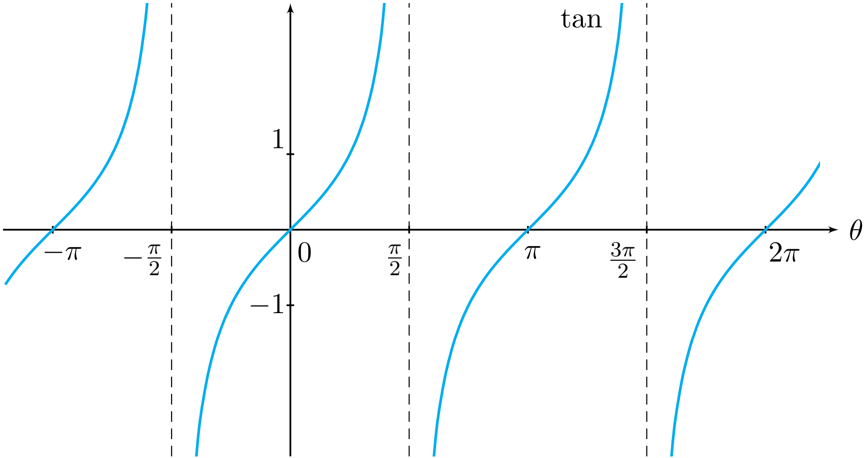


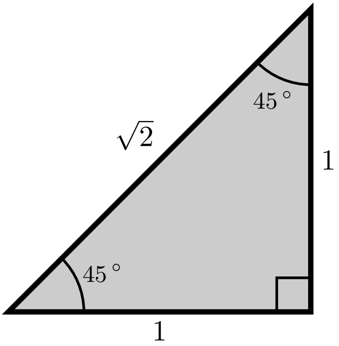
Trigonometry - SinCosTan.nb

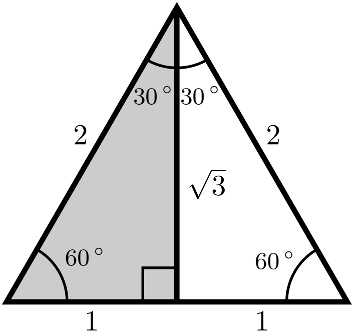

y
sin
1
−1
θ
0
π/2
π
3π/2
2π
5π/2
3π


x
cos
1
−1
θ
0
π/2
π
3π/2
2π
5π/2
3π

Graph of the tangent







$$\cos\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2}$$

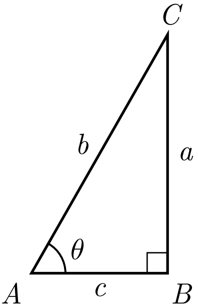
$$\sin\left(\frac{\pi}{4}\right) = \tfrac{1}{2}\sqrt{2}$$

$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$\cos\left(\frac{\pi}{6}\right) = \tfrac{1}{2}\sqrt{3} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \tfrac{1}{2}\sqrt{3}$$

$$\tan\left(\frac{\pi}{6}\right) = \tfrac{1}{3}\sqrt{3} \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$



$$\cos^2\theta + \sin^2\theta =$$

Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ *we have*

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

- **Example**: use $\frac{5}{12}\pi = \dfrac{\pi}{6} + \dfrac{\pi}{4}$ to calculate $\sin\left(\frac{5}{12}\pi\right)$.

Theorem

For arbitrary $\alpha, \beta \in \mathbb{R}$ *we have*

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

- We prove the first equation:

$$\begin{aligned}\sin(\alpha - \beta) &= \sin\left(\alpha + (-\beta)\right)\\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta.\end{aligned}$$

Theorem

For arbitrary $\alpha \in \mathbb{R}$ *we have*

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha.$$

Periodicity	$\sin(\alpha + 2\pi) = \sin\alpha$ and $\sin(\alpha+\pi) = -\sin\alpha$ $\cos(\alpha + 2\pi) = \cos\alpha$ and $\cos(\alpha+\pi) = -\cos\alpha$
Symmetry	$\sin(-\alpha) = -\sin\alpha$ $\cos(-\alpha) = \cos\alpha$
Congruence	$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha$ and $\sin\left(\alpha - \frac{\pi}{2}\right) = -\cos\alpha$ $\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha$ and $\cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha$
Sum formulas	$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Difference formulas	$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
Doubling formulas	$\sin(2\alpha) = 2\sin\alpha\cos\alpha$ $\sin^2\left(\frac{1}{2}\alpha\right) = \frac{1}{2} - \frac{1}{2}\cos\alpha$ $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$ $\cos^2\left(\frac{1}{2}\alpha\right) = \frac{1}{2} + \frac{1}{2}\cos\alpha$
Pythagoras' thm	$\cos^2\alpha + \sin^2\alpha = 1$

(1) a) Use $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ to calculate $\cos\left(\frac{\pi}{12}\right)$.

b) Calculate $\sin\left(\frac{\pi}{12}\right)$.

c) Using the doubling formulas, check the results of a) and b).

(2) Prove the doubling formulas for sine and cosine.